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Abstract

Motivated by the work of Mersini, the particle production related to the tunneling in false vacuum decay is carefully investigated in the thin-wall approximation. It is shown that in this case the particle production is exponentially suppressed even when the momentum is comparable to the curvature scale of the bubble. The number of created particles is ultraviolet finite.

1 General discussion

False vacuum decay by the tunneling in the WKB approximation is described by the $O(4)$ -symmetric bounce solution, $\phi_b(\sqrt{\vec{x}^2 + \tau^2})$. In the limit of small energy-density difference, ϵ , between the true and false vacuum the bounce looks like a large four-dimensional spherical bubble of true vacuum separated by a thin wall from a sea of false vacuum. The evolution of the system after tunneling is described by the $O(3,1)$ invariant solution that is obtained by the analytic continuation of the bounce to the Minkowskian time $\tau \rightarrow it$, see [1]. Since the field ϕ describing the bubble nucleation and expansion is coupled to itself and to the fluctuation field, the false vacuum decay through the barrier penetration is accompanied by the particle production. Using the formalism developed in [2, 3] for the calculation of particle number the particle production by the thin-walled bubble was considered in [4]. However, we fail to corroborate the main conclusion of [4] that for the momentum comparable to the curvature scale of the bubble there is a strong enhancement of the particle production. In this paper we would like to point out that, in general, the particle production in the thin-wall approximation is strongly suppressed. The calculations have been undertaken to exhibit the exponential suppression of the particle production, even when the momentum is comparable to the curvature scale of the bubble, rather than for actual use in calculation of particle number at all momenta. Throughout this paper the metric signature in Minkowski space-time is $(-, +, +, +)$ and units $\hbar = c = 1$ are used. The calculations are present in sec.2.

2 Particle spectrum

Throughout this paper we restrict ourselves to the consideration of thin-wall case. This approximation for the bubble formation is considered in [1]. We consider a scalar field ϕ defined by the Lagrange density

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - U(\phi). \quad (1)$$

The potential has the form

$$U(\phi) = \frac{\lambda}{2}\phi^2(\phi - 2a)^2 - \epsilon(\phi^3/2a^3 - 3\phi^4/16a^4), \quad (2)$$

where λ , a and ϵ are positive parameters. The potential (2) has a local minimum at $\phi_f = 0$, $U(0) = 0$, and a global minimum at $\phi_t = 2a$, $U(2a) = -\epsilon$. Bounce solution with a good accuracy is $\phi_b =$

$a\{1 - \tanh(\mu[\rho - R])\}$ where $\mu = a\sqrt{\lambda}$ and $R = 2\mu^3/\epsilon\lambda$. Thickness of the bubble wall is $L = 2/\mu$. For the potential (2) the condition for the validity of the thin-wall approximation is $R/L = \mu^4/\epsilon\lambda \gg 1$. In other words, in this limiting case the energy-density difference between the true and false vacuum is much smaller than the height of the barrier $U(\phi)$. For the calculation of particle spectrum we use the standard technique developed in [2, 3]. The results relevant for our discussion can be summarized as follows. The coordinate system used in the Euclidean region is $(\rho, \chi, \theta, \varphi)$ where (θ, φ) are usual angle coordinates on two-dimensional sphere and (ρ, χ) are related to $r = |\vec{x}|$ and τ as follows

$$\begin{aligned} r &= \rho \sin(\chi), \quad \tau = -\rho \cos(\chi), \\ 0 &\leq \chi \leq \pi/2, \quad 0 \leq \rho < \infty. \end{aligned} \quad (3)$$

The coordinate system in the Minkowski region may be obtained by replacement $(\rho, \chi) \rightarrow (-i\rho_M, -i\chi_M)$, which yields

$$\begin{aligned} r &= \rho_M \sinh(\chi_M), \quad t = \rho_M \cosh(\chi_M), \\ 0 &< \chi_M < \infty, \quad 0 < \rho_M < \infty. \end{aligned} \quad (4)$$

The basic equation which governs the fluctuation field $\Phi(t, \vec{x})$ reads

$$\left[-\partial_M^2 - \frac{3}{\rho_M} \partial_M + \frac{1}{\rho_M^2} \hat{L}^2 - U''(\phi_b) \right] \Phi = 0, \quad (5)$$

where ∂_M denotes the partial derivative with respect to ρ_M and \hat{L} is the Laplacian operator on three-dimensional unit hyperboloid. Since the bounce solution has $O(3, 1)$ symmetry in Minkowski region one can label Φ by the eigenvalues of the angular momentum, (l, m) as well as by the radial momentum, p , and write

$$\Phi = Y_{plm}(\chi, \theta, \varphi) \frac{\psi_M(\rho_M)}{\rho_M^{3/2}}, \quad (6)$$

where Y_{plm} are eigenfunctions of the operator $-\hat{L}^2$ with eigenvalues $1 + p^2$. After substituting Eq.(6) into Eq.(5) and analytically continuing of Φ to the Euclidean region by replacement $\rho_M \rightarrow i\rho$, $\chi_M \rightarrow -i\chi$ one obtains the equation for the fluctuation field in the under-barrier region

$$\left[\partial_\rho^2 + \frac{(p^2 + 1/4)}{\rho^2} - U''(\phi_b) \right] \psi = 0, \quad (7)$$

where $\psi(\rho) \equiv \psi_M(i\rho)$. This is a zero-energy Schrödinger equation

$$[\partial_\rho^2 - V(\rho)]\psi = 0 \quad (8)$$

of a particle in the potential

$$V(\rho) = -\frac{(p^2 + 1/4)}{\rho^2} + 2\mu^2\{3 \tanh^2(\mu[\rho - R]) - 1\} + \epsilon\text{-term}. \quad (9)$$

Thus, $\psi_M(\rho_M)$ is given by solving Eq.(8) with ρ replaced by $-i\rho$. Since in the thin-wall approximation $\phi_b(0)$ is very close to the true vacuum value $2a$ the solution obtained in this way after the analytical continuation at the turning point $\rho = 0$ will have the form

$$\psi_M(\rho_M) = c_1(p)\sqrt{\rho_M}H_{ip}^{(1)}(\omega\rho_M) + c_2(p)\sqrt{\rho_M}H_{ip}^{(2)}(\omega\rho_M),$$

where $\omega^2 = U''(\phi_b(0))$. At $\tau = -\infty$ the field ϕ is in false vacuum state and correspondingly the fluctuation field Φ satisfies the vanishing boundary condition when $\rho \rightarrow \infty$. The spectrum of created particles $n(p)$ is given by [3]

$$n(p) = \frac{1}{(|c_1(p)/c_2(p)|^2 - 1)}. \quad (10)$$

For detailed description of the general formalism see [2, 3].

If p is of order or smaller than μR one can approximate the potential V as follows

$$V = \begin{cases} 4\mu^2 - v/\rho^2 & \rho_+ < \rho, \\ \tilde{v}_3(x + v_2/v_3)^2 - k & |x| \leq L, \\ 4\mu^2 - v/\rho^2 & \rho < \rho_-, \end{cases} \quad (11)$$

where $v = p^2 + 1/4$, $x = \rho - R$, $v_1 = v/R^2 + 2\mu^2$, $v_2 = v/R^3$, $v_3 = 6\mu^4 - 3v/R^4$, $\tilde{v}_3 = v_3(6 + (p/\mu R)^2)/24$, $k = v_3(v_1/v_3 + v_2^2/v_3^2)$, $\rho_{\pm} = R \pm L$. We have neglected the contribution coming from the ϵ -term because at $\phi = 0$ this term equals zero and at $\phi \sim a$ this is of order $\epsilon\lambda/\mu^2 \ll \mu^2$. The general solution of Eq.(8) in regions $|\rho - R| > L$ is written in terms of modified Bessel functions. For solving this equation we have used the program package Maple 7.

In region $\rho < \rho_-$ the solution reads

$$\psi = a_1\sqrt{\rho}I_{ip}(2\mu\rho) + a_2\sqrt{\rho}K_{ip}(2\mu\rho). \quad (12)$$

In region $|x| \leq L$, the exact solution to this equation is expressed in terms of Whittaker W , M functions [5]. Using the program package Maple 7 one finds that in region $|x| \leq L$ the solution has the form

$$\psi = a_3W(b, 1/4, z^2)/\sqrt{|z|} + a_4M(b, 1/4, z^2)/\sqrt{|z|}, \quad (13)$$

where $z = \tilde{v}_3^{1/4}(x + v_2/v_3)$, $b = k/4\tilde{v}_3^{1/2}$. For $s = p/\mu R$ by taking into account that in the thin-wall approximation $(\mu R)^2 \ll (\mu R)^4$ one gets

$$\begin{aligned} k/v_3^{1/2} &= (v + 2\mu^2 R^2 + v^2/(6\mu^4 R^4 - 3v))/\sqrt{6\mu^4 R^4 - 3v} \approx \\ &((2 + s^2)\mu^2 R^2 + v^2/6\mu^4 R^4)/\sqrt{6\mu^4 R^4} \approx (2 + s^2)/\sqrt{6}, \end{aligned}$$

and

$$\begin{aligned} (\pm 2/\mu + v_2/v_3)v_3^{1/4} &= (\pm 2/\mu R + v/(6\mu^4 R^4 - 3v))(6\mu^4 R^4 - 3v)^{1/4} \approx \\ &(\pm 2/\mu R + v/6\mu^4 R^4)\mu R 6^{1/4} \approx \pm 2 \times 6^{1/4}. \end{aligned}$$

It is convenient to introduce the following notations

$$\begin{aligned} W_{\pm}(s) &= W(b, 1/4, z^2)/\sqrt{|z|}\Big|_{\pm\tilde{z}}, \quad W'_{\pm}(s) = \left\{W(b, 1/4, z^2)/\sqrt{|z|}\right\}'\Big|_{\pm\tilde{z}}, \\ M_{\pm}(s) &= M(b, 1/4, z^2)/\sqrt{|z|}\Big|_{\pm\tilde{z}}, \quad M'_{\pm}(s) = \left\{M(b, 1/4, z^2)/\sqrt{|z|}\right\}'\Big|_{\pm\tilde{z}}, \end{aligned}$$

where $\pm\tilde{z} = z(x = \pm L) \approx \pm 0.1304 \times (6 + s^2)^{1/4}$ and the prime denotes differentiation with respect to z . These quantities are needed for the matchings at ρ_{\pm} . Using the program package Maple 7 one can estimate W_{\pm} , W'_{\pm} , M_{\pm} , M'_{\pm} at the desired value of s .

In region $\rho > \rho_+$ the solution satisfying the vanishing boundary condition when $\rho \rightarrow \infty$ reads

$$\psi = a_5\sqrt{\rho}K_{ip}(2\mu\rho). \quad (14)$$

For our purposes the quantity of immediate interest is $n(p) = |a_1(p)/a_2(p)\pi|^2$, see [3]. So, one can omit the common multiplier of a_1, a_2 coefficients as well as the common multiplier of a_3, a_4 coefficients. The matching at ρ_+ gives

$$\begin{aligned} a_3 &\sim M'_+ K_{ip}(2\mu\rho_+) - M_+ \{K_{ip}(2\mu\rho_+)/2\tilde{v}_3^{1/4}\rho_+ + K'_{ip}(2\mu\rho_+)2\mu/\tilde{v}_3^{1/4}\}, \\ a_4 &\sim W_+ \{K_{ip}(2\mu\rho_+)/2\tilde{v}_3^{1/4}\rho_+ + K'_{ip}(2\mu\rho_+)2\mu/\tilde{v}_3^{1/4}\} - W'_+ K_{ip}(2\mu\rho_+), \end{aligned} \quad (15)$$

the proportionality coefficients are the same for a_3 and a_4 , here and below the prime denotes differentiation with respect to the whole argument. The matching at ρ_- gives

$$\begin{aligned} a_1 &\sim \{K_{ip}(2\mu\rho_-)/2\tilde{v}_3^{1/4}\rho_- + K'_{ip}(2\mu\rho_-)2\mu/\tilde{v}_3^{1/4}\}\{a_3W_- + a_4M_-\} - \\ &\quad K_{ip}(2\mu\rho_-)\{a_3W'_- + a_4M'_-\}, \\ a_2 &\sim -\{I_{ip}(2\mu\rho_-)/2\tilde{v}_3^{1/4}\rho_- + I'_{ip}(2\mu\rho_-)2\mu/\tilde{v}_3^{1/4}\}\{a_3W_- + a_4M_-\} + \\ &\quad I_{ip}(2\mu\rho_-)\{a_3W'_- + a_4M'_-\}, \end{aligned} \quad (16)$$

the proportionality coefficients are the same for a_1 and a_2 . In the thin-wall approximation $\mu\rho_\pm \gg 1$ and for evaluating the modified Bessel functions of the $2\mu\rho_\pm$'s one can use the asymptotic formulae. For $p \ll \sqrt{\mu R}$ one can use the following asymptotic expansions, see [6],

$$\begin{aligned} I_{ip}(x) &\sim \exp(x)\{1 + (4p^2 + 1)/8x + \dots\}/\sqrt{2\pi x}, \\ I'_{ip}(x) &\sim \exp(x)\{1 - (3 - 4p^2)/8x + \dots\}/\sqrt{2\pi x}, \\ K_{ip}(x) &\sim \exp(-x)\{1 - (4p^2 - 1)/8x + \dots\}\sqrt{\pi/2x}, \\ K'_{ip}(x) &\sim -\exp(-x)\{1 + (3 - 4p^2)/8x + \dots\}\sqrt{\pi/2x}. \end{aligned} \quad (17)$$

For $p \sim \mu R$ the asymptotic expansions of modified Bessel functions of purely imaginary order and their derivatives are derived in [7]. Motivated in part by the fact that $K_{ip}(x)$, where x is real variable, is the imaginary part of $I_{ip}(x)$ up to a multiplicative factor, recent investigations have been carried out on the real part $I_{ip}(x)$, denoted by $L_{ip}(x)$,

$$L_{ip}(x) = \frac{1}{2}\{I_{ip}(x) + I_{-ip}(x)\}.$$

Any of $L_{ip}(x)$, $K_{ip}(x)$ and $I_{ip}(x)$ can be constructed from the remaining two functions by the identity

$$I_{ip}(x) = L_{ip}(x) - i\frac{\sinh(p\pi)}{\pi}K_{ip}(x). \quad (18)$$

Both $K_{ip}(x)$ and $L_{ip}(x)$ are real valued and even functions of p .

For $p < x$, introducing $p = x \sin \theta$, where $0 < \theta < \pi/2$, the leading terms of asymptotic expansions are [7]

$$\begin{aligned} L_{ip}(p \csc \theta) &\sim \exp(p[\cot \theta + \theta])/\sqrt{2p\pi \cot \theta} + \exp(p[\pi - \theta - \cot \theta])3p \cot^2 \theta/2\pi, \\ L'_{ip}(p \csc \theta) &\sim \exp(p[\cot \theta + \theta])\sqrt{\cot \theta/2p\pi} + \exp(p[\pi - \theta - \cot \theta])\tan \theta \sin \theta/3\pi, \\ K_{ip}(p \csc \theta) &\sim \exp(-p[\cot \theta + \theta])\sqrt{\pi/2p \cot \theta}, \\ K'_{ip}(p \csc \theta) &\sim -\exp(-p[\cot \theta + \theta])\sin \theta \sqrt{\pi \cot \theta/2p}. \end{aligned} \quad (19)$$

For $p \approx x$ the leading terms of asymptotic expansions take the form [7]

$$\begin{aligned} L_{ip}(p) &\sim \exp(p\pi/2)\{Bi(0)/2^{2/3}p^{1/3} + Bi'(0)/2^{1/3}p^{5/3}70\}, \\ L'_{ip}(p) &\sim \exp(p\pi/2)\{Bi'(0)/2^{1/3}p^{2/3} - Bi(0)/2^{2/3}p^{4/3}5\}, \\ K_{ip}(p) &\sim \pi \exp(-p\pi/2)\{2^{1/3}Ai(0)/p^{1/3} + 2^{2/3}Ai'(0)/70p^{5/3}\}, \\ K'_{ip}(p) &\sim \pi \exp(-p\pi/2)\{2^{2/3}Ai'(0)/p^{2/3} - 2^{1/3}Ai(0)/5p^{4/3}\}, \end{aligned} \quad (20)$$

where Ai , Bi are Airy functions.

For $p > x$, introducing $p = x \cosh \nu$ and $\alpha = p(\tanh \nu - \nu) + \pi/4$, where $\nu > 0$, the leading terms of asymptotic expansions are [7]

$$\begin{aligned} L_{ip}(p \operatorname{sech} \nu) &\sim \exp(p\pi/2) \sqrt{\coth \nu} \{\cos \alpha + \sin \alpha\} / \sqrt{2p\pi}, \\ L'_{ip}(p \operatorname{sech} \nu) &\sim \exp(p\pi/2) \cosh \nu \{\sin \alpha - \cos \alpha\} / \sqrt{2p\pi \coth \nu}, \\ K_{ip}(p \operatorname{sech} \nu) &\sim \exp(-p\pi/2) \sqrt{2\pi \coth \nu} \{\cos \alpha - \sin \alpha\} / \sqrt{p}, \\ K'_{ip}(p \operatorname{sech} \nu) &\sim -\exp(-p\pi/2) \sqrt{2\pi} \cosh \nu \{\sin \alpha + \cos \alpha\} / \sqrt{p \coth \nu}. \end{aligned} \quad (21)$$

We can now analyze the particle spectrum. Using the leading asymptotic terms from Eq.(17) for small values of p one gets

$$n(p) \approx 16(\mu R)^2 \exp(-8\mu R). \quad (22)$$

Notice that this result is different from that one obtained in [4] for small values of p . Taking into account that W_{\pm} , M_{\pm} , W'_{\pm} , M'_{\pm} are real valued functions from Eqs.(15,16,18) one obtains

$$a_2 \sim \operatorname{Re}(a_2) + i \frac{\sinh(p\pi)}{\pi} a_1,$$

and correspondingly

$$n(p) = \frac{1}{(\pi \operatorname{Re}(a_2)/a_1)^2 + \sinh^2(p\pi)}. \quad (23)$$

Thus, If $p \gg 1$ from Eq.(23) one simply concludes that $n(p)$ is suppressed at least as

$$n(p) \sim \exp(-2p\pi). \quad (24)$$

As it is clearly seen from Eq.(23) $n(p)$ vanishes for that values of p at which $\operatorname{Re}(a_2)/a_1$ becomes infinity. Using Eqs.(19,20,21) one can easily estimate the term $\operatorname{Re}(a_2)/a_1$ with exponential accuracy. If p is large, of order μR , but small than $2\mu R$ such that $\arcsin(p/x)$ is not very close to $\pi/2$ from Eq.(19) one gets

$$\operatorname{Re}(a_2)/a_1 \sim \exp(4p[\cot \theta + \theta]), \quad (25)$$

where $0 < \theta < \pi/2$. For the values of θ very close to $\pi/2$ one has to use Eq.(20). If p is close to $2\mu R$ using Eq.(20) one gets

$$\operatorname{Re}(a_2)/a_1 \sim \exp(4\pi\mu R). \quad (26)$$

For p of order μR such that $p > 2\mu R$ from Eq.(21) one gets

$$\operatorname{Re}(a_2)/a_1 \sim \exp(2p\pi). \quad (27)$$

For very large momentum, assuming for instance $p \gg (\mu R)^2$, one can approximate $V(\rho)$ as follows

$$V = \begin{cases} 4\mu^2 - v/\rho^2 & \rho_+ \leq \rho, \\ -2\mu^2 - v/\rho^2 & |x| < L, \\ 4\mu^2 - v/\rho^2 & \rho \leq \rho_-. \end{cases} \quad (28)$$

Namely, we have replaced the well of the second term of Eq.(9) in region $|x| < L$ by the square well, and if p is very large one can neglect the contribution coming from such a deformation of the potential in region $|x| < L$ in comparison with the term v/ρ^2 . Now in region $|x| < L$ the general solution of Eq.(8) is expressed in terms of Bessel functions of purely imaginary order. With no loss of generality one can choose the solution in this region to be real valued function. For instance in this region one can take

$$\psi = a_3 \sqrt{\rho} \operatorname{Re}(J_{ip}(\sqrt{2}\mu\rho)) + a_4 \sqrt{\rho} \operatorname{Re}(Y_{ip}(\sqrt{2}\mu\rho)). \quad (29)$$

It is clear that for the particle spectrum one gets again the Eq.(23) and simply concludes that for very large momentum $n(p)$ is suppressed at least as

$$n(p) \sim \exp(-2p\pi). \quad (30)$$

The asymptotic behavior of Bessel functions of large purely imaginary order can be found in [8]. Namely, in [8] one can find the asymptotic expansions of $J_{ip}(x)$ and $H_{ip}^{(1)}(x)$,

$$\begin{aligned} J_{ip}(x) &\sim \exp(p\pi/2) \exp(i\sqrt{p^2+x^2} - ip \operatorname{arsh}(p/x) - i\pi/4) / \sqrt{2\pi}(p^2-x^2)^{1/4}, \\ H_{ip}^{(1)}(x) &\sim \sqrt{2} \exp(p\pi/2) \exp(i\sqrt{p^2+x^2} - ip \operatorname{arsh}(p/x) - i\pi/4) / \pi(p^2+x^2)^{1/4}, \end{aligned} \quad (31)$$

and using the standard relation $Y_{ip}(x) = i\{J_{ip}(x) - H_{ip}^{(1)}(x)\}$ can determine the asymptotic behavior of $Y_{ip}(x)$. Using Eqs.(21,31) one can easily estimate the ratio $Re(a_2)/a_1$ with exponential accuracy. The matchings at ρ_+ , ρ_- give $a_{3,4} \sim \exp(-p\pi)$ and $a_1 \sim \exp(-p\pi/2)$, $a_2 \sim \exp(p\pi/2)$ respectively. Correspondingly, for very large momentum

$$Re(a_2)/a_1 \sim \exp(2p\pi). \quad (32)$$

Thus, we have explicitly shown that, in general, the particle production in the thin-wall approximation is exponentially suppressed. Using Eqs.(21,31) one can also evaluate the ultraviolet finiteness of the number of created particles. Without carrying out the explicit computation from these asymptotic formulae one concludes that for very large momentum the particle production is controlled by the term $n(p) \sim \exp(-2p\pi) \times (\text{rational function of } p)$. As an unessential to this problem we do not take into account the terms enclosed in the braces in Eq.(21). For this expression the integral $\int_{p \gg (\mu R)^2}^{\infty} n(p)$ is certainly convergent. So we infer that the number of created particles is ultraviolet finite. This completes the review.

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